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EXTENSION OF EUCLIDEAN OPERATOR RADIUS INEQUALITIES

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Abstract. To extend the Euclidean operator radius, we define w_p for an n-tuples of operators $(T_1,\dots,T_n)$ in $\mathbb{B}(\mathscr{H})$ by $w_p(T_1,\dots,T_n) := \sup_{\|x\|=1}\left(\sum_{i=1}^n |\langle T_i x, x\rangle|^p\right)^{\frac{1}{p}}$ for $p\geq 1$. We generalize some inequalities including Euclidean operator radius of two operators to those involving w_p. Further we obtain some lower and upper bounds for w_p. Our main result states that if f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)\,g(t)=t$ for all $t\in[0,\infty)$, then

$$w_p^{rp}\left(A_1^*T_1B_1,\dots,A_n^*T_nB_n\right)\leq\frac{1}{2}\left\|\sum_{i=1}^n\Big(\left[B_i^*f^2\left(|T_i|\right)B_i\right]^{rp}+\left[A_i^*g^2\left(|T_i^*|\right)A_i\right]^{rp}\Big)\right\|$$

for all $p\geq 1$, $r\geq 1$ and operators in $\mathbb{B}(\mathscr{H})$.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$. The numerical radius of $A\in\mathbb{B}(\mathscr{H})$ is defined by

$$w(A)=\sup\{|\langle Ax,x\rangle| : x\in\mathscr{H}, \|x\|=1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. Namely, we have

$$\frac{1}{2}\|A\|\leq w(A)\leq\|A\|.$$

for each $A\in\mathbb{B}(\mathscr{H})$. It is known that if $A\in\mathbb{B}(\mathscr{H})$ is self-adjoint, then $w(A)=\|A\|$. An important inequality for $w(A)$ is the power inequality stating that $w(A^n)\leq w^n(A)$ for $n=1,2,\dots$. There are many inequalities involving numerical radius; see [2, 3, 4, 10, 11, 12] and references therein.

The Euclidean operator radius of an n-tuple $(T_1,\dots,T_n)\in\mathbb{B}(\mathscr{H})^{(n)} := \mathbb{B}(\mathscr{H})\times$

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$\ldots \times \mathbb{B}(\mathscr{H})$ was defined in [9] by

$$w_e(T_1,\ldots,T_n) := \sup_{\|x\|=1}\left(\sum_{i=1}^{n}|\langle T_ix,x\rangle|^2\right)^{\frac{1}{2}}.$$

The particular cases $n=1$ and $n=2$ are numerical radius and Euclidean operator radius. Some interesting properties of this radius were obtained in [9]. For example, it is established that

$$\frac{1}{2\sqrt{n}}\left\|\sum_{i=1}^{n}T_iT_i^*\right\|^{\frac{1}{2}} \leq w_e(T_1,\cdots,T_n) \leq \left\|\sum_{i=1}^{n}T_iT_i^*\right\|^{\frac{1}{2}}. \tag{1.1}$$

We also observe that if $A=B+iC$ is the Cartesian decomposition of A, then

$$w_e^2(B,C) = \sup_{\|x\|=1}\{|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2\} = \sup_{\|x\|=1}|\langle Ax,x\rangle|^2 = w^2(A).$$

By the above inequality and $A^*A+AA^*=2(B^2+C^2)$, we have

$$\frac{1}{16}\|A^*A+AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A+AA^*\|.$$

We define w_p for n-tuples of operators $(T_1,\ldots,T_n)\in\mathbb{B}(\mathscr{H})^{(n)}$ for $p\geq 1$ by

$$w_p(T_1,\ldots,T_n) := \sup_{\|x\|=1}\left(\sum_{i=1}^{n}|\langle T_ix,x\rangle|^p\right)^{\frac{1}{p}}.$$

It follows from Minkowski's inequality for two vectors $a=(a_1,a_2)$ and $b=(b_1,b_2)$, namely,

$$(|a_1+b_1|^p+|a_2+b_2|^p)^{\frac{1}{p}} \leq (|a_1|^p+|a_2|^p)^{\frac{1}{p}}+(|b_1|^p+|b_2|^p)^{\frac{1}{p}} \quad \text{for} \quad p>1$$

that w_p is a norm.

Moreover $w_p, p\geq 1$, for n-tuple of operators $(T_1,\ldots,T_n)\in\mathbb{B}(\mathscr{H})^{(n)}$ satisfies the following properties:

(i) $w_p\left(T_1,\ldots,T_n\right)=0 \Leftrightarrow T_1=\ldots=T_n=0$.

(ii) $w_p\left(\lambda T_1,\ldots,\lambda T_n\right)=|\lambda|\,w_p\left(T_1,\ldots,T_n\right)$ for all $\lambda\in\mathbb{C}$.

(iii) $w_p\left(T_1+T_1',\ldots,T_n+T_n'\right)\leq w_p\left(T_1,\ldots,T_n\right)+w_p\left(T_1',\ldots,T_n'\right)$ for $(T_1',\ldots,T_n')\in\mathbb{B}(\mathscr{H})^{(n)}$.

(iv) $w_p\left(X^*T_1X,\ldots,X^*T_nX\right)\leq\|X\|^2\,w_p\left(T_1,\ldots,T_n\right)$ for $X\in\mathbb{B}(\mathscr{H})$.

Dragomir [1] obtained some inequalities for the Euclidean operator radius $w_e(B,C)=\sup_{\|x\|=1}\left(|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2\right)^{\frac{1}{2}}$ of two bounded linear operators in a Hilbert space. In section 2 of this paper we extend some his results including inequalities for the Euclidean operator radius of linear operators to w_p $(p\geq 1)$. In addition, we apply some known inequalities for getting new inequalities for w_p in two operators.

In section 3 we prove inequalities for w_p for n-tuples of operators. Some of our result in this section, generalize some inequalities in section 2. Further, we find some lower and upper bounds for w_p.

2. Inequalities for w_p for two operators

To prove our generalized numerical radius inequalities, we need several known lemmas. The first lemma is a simple result of the classical Jensen inequality and a generalized mixed Cauchy–Schwarz inequality [7, 8, 6].

Lemma 2.1. *For $a, b \geq 0$, $0 \leq \alpha \leq 1$ and $r \neq 0$,*

(a) $a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq [\alpha a^r + (1-\alpha)b^r]^{\frac{1}{r}}$ *for* $r \geq 1$,

(b) *If $A \in \mathbb{B}(\mathscr{H})$, then $|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha}x, x\rangle \langle |A^*|^{2(1-\alpha)}y, y\rangle$ for all $x, y \in \mathscr{H}$, where $|A| = (A^*A)^{\frac{1}{2}}$.*

(c) *Let $A \in \mathbb{B}(\mathscr{H})$, and f and g be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)\, g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)\, x\| \, \|g(|A^*|)\, y\|$$

for all $x, y \in \mathscr{H}$.

Lemma 2.2 (**McCarthy inequality** [5])**.** *Let $A \in \mathbb{B}(\mathscr{H})$, $A \geq 0$ and let $x \in \mathscr{H}$ be any unit vector. Then*

(a) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ *for* $r \geq 1$,

(b) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for* $0 < r \leq 1$.

Inequalities of the following lemma were obtained for the first time by Clarkson[7].

Lemma 2.3. *Let X be a normed space and $x, y \in X$. Then for all $p \geq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

(a) $2(\|x\|^p + \|y\|^p)^{q-1} \leq \|x+y\|^q + \|x-y\|^q$,

(b) $2(\|x\|^p + \|y\|^p) \leq \|x+y\|^p + \|x-y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p)$,

(c) $\|x+y\|^p + \|x-y\|^p \leq 2(\|x\|^q + \|y\|^q)^{p-1}$.

If $1 < p \leq 2$ the converse inequalities hold.

Making the transformations $x \to \frac{x+y}{2}$ and $y \to \frac{x-y}{2}$ we observe that inequalities (a) and (c) in Lemma 2.3 are equivalent and so are the first and the second inequalities of (b). First of all we obtain a relation between w_p and w_e for $p \geq 1$.

Proposition 2.4. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then*

$$w_p(B,C) \leq w_q(B,C) \leq 2^{\frac{1}{q}-\frac{1}{p}} w_p(B,C)$$

for $p \geq q \geq 1$. In particular

$$w_p(B,C) \leq w_e(B,C) \leq 2^{\frac{1}{2}-\frac{1}{p}} w_p(B,C) \tag{2.1}$$

for $p \geq 2$, and

$$2^{\frac{1}{2}-\frac{1}{p}} w_p(B,C) \leq w_e(B,C) \leq w_p(B,C)$$

for $1 \leq p \leq 2$.

Proof. An application of Jensen's inequality says that for $a, b > 0$ and $p \geq q > 0$, we have

$$(a^p + b^p)^{\frac{1}{p}} \leq (a^q + b^q)^{\frac{1}{q}}.$$

Let $x \in \mathscr{H}$ be a unit vector. Choosing $a = |\langle Bx, x\rangle|$ and $b = |\langle Cx, x\rangle|$, we have

$$\Big(|\langle Bx, x\rangle|^p + |\langle Cx, x\rangle|^p \Big)^{\frac{1}{p}} \leq \Big(|\langle Bx, x\rangle|^q + |\langle Cx, x\rangle|^q \Big)^{\frac{1}{q}}.$$

Now the first inequality follows by taking the supremum over all unit vectors in $\mathscr{H}$. A simple consequence of the classical Jensen's inequality concerning the convexity or the concavity of certain power functions says that for $a, b \geq 0, 0 \leq \alpha \leq 1$ and $p \geq q$, we have

$$(\alpha a^q + (1-\alpha) b^q)^{\frac{1}{q}} \leq (\alpha a^p + (1-\alpha) b^p)^{\frac{1}{p}}.$$

For $\alpha = \frac{1}{2}$, we get

$$(a^q + b^q)^{\frac{1}{q}} \leq 2^{\frac{1}{q}-\frac{1}{p}} (a^p + b^p)^{\frac{1}{p}}.$$

Again let $x \in \mathscr{H}$ be a unit vector. Choosing $a = |\langle Bx, x\rangle|$ and $b = |\langle Cx, x\rangle|$ we get

$$\Big(|\langle Bx, x\rangle|^q + |\langle Cx, x\rangle|^q \Big)^{\frac{1}{q}} \leq 2^{\frac{1}{q}-\frac{1}{p}} \Big(|\langle Bx, x\rangle|^p + |\langle Cx, x\rangle|^p \Big)^{\frac{1}{p}}.$$

Now the second inequality follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

On making use of inequality (2.1) we find a lower bound for w_p $(p \geq 2)$.

Corollary 2.5. *If $B, C \in \mathbb{B}(\mathscr{H})$, then for $p \geq 2$*

$$w_p(B,C) \geq 2^{\frac{1}{p}-2} \|B^*B + C^*C\|^{\frac{1}{2}}.$$

Proof. According to inequalities (1.1) and (2.1) we can write

$$w_e(B,C) \geq \frac{1}{2\sqrt{2}}\|B^*B + C^*C\|^{\frac{1}{2}}$$

and

$$w_p(B,C) \geq 2^{\frac{1}{p}-\frac{1}{2}} w_e(B,C),$$

respectively. We therefore get desired inequality. □

The next result is concerned with some lower bounds for w_p. This consequence has several inequalities as special cases. Our result will be generalized to n-tuples of operators in the next section.

Proposition 2.6. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then for $p \geq 1$*

$$w_p(B,C) \geq 2^{\frac{1}{p}-1} \max\left(w(B+C), w(B-C)\right). \tag{2.2}$$

This inequality is sharp.

Proof. We use convexity of function $f(t) = t^p$ $(p \geq 1)$ as follows:

$$\begin{aligned}(|\langle Bx, x\rangle|^p + |\langle Cx, x\rangle|^p)^{\frac{1}{p}} &\geq 2^{\frac{1}{p}-1}(|\langle Bx, x\rangle| + |\langle Cx, x\rangle|)\\ &\geq 2^{\frac{1}{p}-1}|\langle Bx, x\rangle \pm \langle Cx, x\rangle|\\ &= 2^{\frac{1}{p}-1}|\langle (B \pm C)x, x\rangle|\,.\end{aligned}$$

Taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ yields that

$$w_p(B,C) \geq 2^{\frac{1}{p}-1} w(B \pm C).$$

For sharpness one can obtain the same quantity $2^{\frac{1}{p}} w(B)$ on both sides of the inequality by putting $B = C$. □

Corollary 2.7. *If $A = B + iC$ is the Cartesian decomposition of A, then for all $p \geq 2$*

$$w_p(B,C) \geq 2^{\frac{1}{p}-1} \max\left(\|B+C\|, \|B-C\|\right),$$

and

$$w(A) \geq 2^{\frac{1}{p}-2} \max\left(\|(1-i)A + (1+i)A^*\|, \|(1+i)A + (1-i)A^*\|\right)$$

Proof. Obviously by inequality (2.2) we have the first inequality. For the second we use inequality (2.1). □

Corollary 2.8. *If $B, C \in \mathbb{B}(\mathscr{H})$, then for $p \geq 1$*

$$w_p(B,C) \geq 2^{\frac{1}{p}-1} \max\{w(B), w(C)\}. \tag{2.3}$$

In addition, if $A = B + iC$ is the Cartesian decomposition of A, then for $p \geq 2$

$$w(A) \geq 2^{\frac{1}{p}-2} \max\left(\|A + A^*\|, \|A - A^*\|\right).$$

Proof. By inequality (2.2) and properties of the numerical radius, we have

$$2w_p(B,C) \geq 2^{\frac{1}{p}-1}(w(B+C) + w(B-C)) \geq 2^{\frac{1}{p}-1} w(B+C+B-C).$$

So

$$w_p(B,C) \geq 2^{\frac{1}{p}-1} w(B).$$

By symmetry we conclude that

$$w_p(B,C) \geq 2^{\frac{1}{p}-1} \max(w(B), w(C)).$$

While the second inequality follows easily from inequality (2.1). □

Now we apply part (b) of Lemma 2.3 to find some lower and upper bounds for w_p $(p > 1)$.

Proposition 2.9. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then for all $p \geq 2$,*
(i) $2^{\frac{1}{p}-1} w_p(B+C, B-C) \leq w_p(B,C) \leq 2^{-\frac{1}{p}} w_p(B+C, B-C)$;
(ii) $2^{\frac{1}{p}-1}\left(w^p(B+C) + w^p(B-C)\right)^{\frac{1}{p}} \leq w_p(B,C) \leq 2^{-\frac{1}{p}}\left(w^p(B+C) + w^p(B-C)\right)^{\frac{1}{p}}$.
If $1 < p \leq 2$ these inequalities hold in the opposite direction.

Proof. Let $x \in \mathscr{H}$ be a unit vector. Part (b) of Lemma 2.3 implies that for any $p \geq 2$

$$2^{1-p}(|a+b|^p + |a-b|^p) \leq |a|^p + |b|^p \leq \frac{1}{2}(|a+b|^p + |a-b|^p).$$

Replacing $a = |\langle Bx, x\rangle|$ and $b = |\langle Cx, x\rangle|$ in above inequalities we obtain the desired inequalities. □

Remark 2.10. In inequality (2.3), if we take $B + C$ and $B - C$ instead of B and C, then for $p \geq 1$

$$w_p(B+C, B-C) \geq 2^{\frac{1}{p}-1} \max\{w(B+C), w(B-C)\}.$$

By employing the first inequality of part (i) of Proposition 2.9, we get

$$w_p(B,C) \geq 2^{\frac{2}{p}-2} \max\{w(B+C), w(B-C)\}$$

for $p \geq 1$.

Taking $B+C$ and $B-C$ instead of B and C in the second inequality of part (ii) of Proposition 2.9, we reach

$$w_p(B+C, B-C) \leq 2^{1-\frac{1}{p}} \left(w^p(B) + w^p(C)\right)^{\frac{1}{p}} .$$

for all $p \geq 1$.

Now by applying the second inequality of part (i) of Proposition 2.9, we infer for $p \geq 1$ that

$$w_p(B,C) \leq 2^{1-\frac{2}{p}} \left(w^p(B) + w^p(C)\right)^{\frac{1}{p}} .$$

So

$$2^{\frac{2}{p}-2} \max\{w(B+C), w(B-C)\} \leq w_p(B,C) \leq 2^{1-\frac{2}{p}} \left(w^p(B) + w^p(C)\right)^{\frac{1}{p}} .$$

Moreover if B and C are self-adjoint, then

$$2^{\frac{2}{p}-2} \max\{\|B+C\|, \|B-C\|\} \leq w_p(B,C) \leq 2^{1-\frac{2}{p}} \left(\|B\|^p + \|C\|^p\right)^{\frac{1}{p}}$$

for all $p \geq 1$.

In the following result we find another lower bound for w_p $(p \geq 1)$.

Theorem 2.11. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then for $p \geq 1$*

$$w_p(B,C) \geq 2^{\frac{1}{p}-1} w^{\frac{1}{2}}(B^2 + C^2).$$

Proof. It follows from (2.2) that

$$2^{\frac{2}{p}-2} w^2(B \pm C) \leq w_p^2(B,C) .$$

Hence

$$\begin{aligned} 2w_p^2(B,C) &\geq 2^{\frac{2}{p}-2} \Big[w^2(B+C) + w^2(B-C)\Big] \\ &\geq 2^{\frac{2}{p}-2} \Big[w\left((B+C)^2\right) + w\left((B-C)^2\right)\Big] \\ &\geq 2^{\frac{2}{p}-2} \Big[w\left((B+C)^2 + (B-C)^2\right)\Big] = 2^{\frac{2}{p}-1} w(B^2 + C^2) . \end{aligned}$$

It follows that

$$w_p(B,C) \geq 2^{\frac{1}{p}-1} w^{\frac{1}{2}}(B^2 + C^2).$$

$\square$

Corollary 2.12. *If $A = B + iC$ is the Cartesian decomposition of A , then*

$$w_p(B, C) \geq 2^{\frac{1}{p}-1} \|B^2 + C^2\|^{\frac{1}{2}}.$$

And

$$w(A) \geq 2^{\frac{1}{p}-\frac{3}{2}} \|A^*A + AA^*\|^{\frac{1}{2}}.$$

for any $p \geq 2$.

Proof. The first inequality is obvious. For the second we have $A^*A + AA^* = 2(B^2 + C^2)$. Now by using inequality (2.1) the proof is complete. □

Corollary 2.13. *If $B, C \in \mathbb{B}(\mathscr{H})$, then for $p \geq 2$*

$$w_p(B, C) \geq 2^{\frac{2}{p}-\frac{3}{2}} w^{\frac{1}{2}} \left(B^2 + C^2\right).$$

Proof. By choosing $B + C$ and $B - C$ instead of B and C in Theorem 2.11 and employing part (i) of Proposition 2.9 we conclude that the desired inequality. □

The following result providing other bound for w_p $(p > 1)$ may be stated as follows:

Proposition 2.14. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then*

$$w_p(B, C) \leq w_q \left(\frac{B + C}{2}, \frac{B - C}{2} \right).$$

for any $p \geq 2, 1 < q \leq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $1 < p \leq 2$, the reverse inequality holds.

Proof. Let $x \in \mathscr{H}$ be a unit vector. Part (a) of Lemma 2.3 implies that

$$|a|^p + |b|^p \leq 2^{\frac{1}{1-q}} \left(|a + b|^q + |a - b|^q\right)^{\frac{1}{q-1}} .$$

So

$$\left(|a|^p + |b|^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p(1-q)}} \left(|a + b|^q + |a - b|^q\right)^{\frac{1}{p(q-1)}} .$$

Now replacing $a = \langle Bx, x\rangle$ and $b = \langle Cx, x\rangle$ in the above inequality we conclude that

$$\left(|\langle Bx, x\rangle|^p + |\langle Cx, x\rangle|^p\right)^{\frac{1}{p}} \leq \left(\left| \left\langle \left(\frac{B + C}{2} \right) x, x \right\rangle \right|^q + \left| \left\langle \left(\frac{B - C}{2} \right) x, x \right\rangle \right|^q \right)^{\frac{1}{q}}. \tag{2.4}$$

By taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ we deduce that

$$w_p(B, C) \leq w_q \left(\frac{B + C}{2}, \frac{B - C}{2} \right)$$

for any $p \geq 2, 1 < q \leq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. □

Corollary 2.15. *Inequality (2.4) implies that*

$$w_p(B,C) \leq \left(w^q \left(\frac{B+C}{2} \right) + w^q \left(\frac{B-C}{2} \right) \right)^{\frac{1}{q}}.$$

for any $1 < q \leq 2, p \geq 2$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Further, if* B *and* C *are self-adjoint, then*

$$w_p(B,C) \leq \frac{1}{2} \left(\|B+C\|^q + \|B-C\|^q \right)^{\frac{1}{q}}.$$

If $1 < p \leq 2$, *the converse inequalities hold.*

Corollary 2.16. *If* $B, C \in \mathbb{B}(\mathscr{H})$, *then*

$$w_q \left(\frac{B+C}{2}, \frac{B-C}{2} \right) \leq 2^{\frac{1}{p}} w_p \left(\frac{B+C}{2}, \frac{B-C}{2} \right).$$

for all $1 < p \leq 2$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $p \geq 2$, *the above inequality is valid in the opposite direction.*

Proof. By Proposition 2.14 we have

$$w_q \left(\frac{B+C}{2}, \frac{B-C}{2} \right) \leq w_p(B,C).$$

for all $1 < p \leq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. Proposition 2.9 follows that

$$w_p(B,C) \leq 2^{\frac{1}{p}-1} w_p(B+C, B-C) = 2^{\frac{1}{p}} w_p \left(\frac{B+C}{2}, \frac{B-C}{2} \right).$$

We therefore get the desired inequality. □

3. Inequalities of w_p for n-tuples of operators

In this section, we are going to obtain some numerical radius inequalities for n-tuples of operators. Some generalization of inequalities in the previous section are also established. According to the definition of numerical radius, we immediately get the following double inequality for $p \geq 1$

$$w_p(T_1, \ldots, T_n) \leq \left(\sum_{i=1}^{n} w^p(T_i) \right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} w(T_i).$$

An application of Holder's inequality gives the next result, which is a generalization of inequality (2.2).

Theorem 3.1. *Let* $(T_1, \ldots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$ *and* $0 \leq \alpha_i \leq 1$*,* $i = 1, \ldots n$*, with* $\sum_{i=1}^{n} \alpha_i = 1$*. Then*

$$w_p\left(T_1, \ldots, T_n\right) \geq w\left(\alpha_1^{1-\frac{1}{p}} T_1 \pm \alpha_2^{1-\frac{1}{p}} T_2 \pm \ldots \pm \alpha_n^{1-\frac{1}{p}} T_n\right)$$

for any $p > 1$*.*

Proof. In the Euclidean space $\mathbb{R}^n$ with the standard inner product, Holder's inequality

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

holds, where p and q are in the open interval $(1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in \mathbb{R}^n$. For $(y_1, \ldots, y_n) = \left(\alpha_1^{1-\frac{1}{p}}, \ldots, \alpha_n^{1-\frac{1}{p}}\right)$ we have

$$\sum_{i=1}^{n} \left|\alpha_i^{1-\frac{1}{p}} x_i\right| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \left|\alpha_i^{1-\frac{1}{p}}\right|^q\right)^{\frac{1}{q}}.$$

Thus

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \geq \sum_{i=1}^{n} \left|\alpha_i^{1-\frac{1}{p}} x_i\right|.$$

Choosing $x_i = |\langle T_i x, x\rangle|$, $i = 1, \ldots n$, we get

$$\begin{aligned}
&\left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p\right)^{\frac{1}{p}} \\
&\quad \geq \sum_{i=1}^{n} \left|\left\langle \alpha_i^{1-\frac{1}{p}} T_i x, x\right\rangle\right| \\
&\quad \geq \left|\left\langle \alpha_1^{1-\frac{1}{p}} T_1 x, x\right\rangle \pm \left\langle \alpha_2^{1-\frac{1}{p}} T_2 x, x\right\rangle \pm \ldots \pm \left\langle \alpha_n^{1-\frac{1}{p}} T_n x, x\right\rangle\right| \\
&\quad = \left|\left\langle \left(\alpha_1^{1-\frac{1}{p}} T_1 \pm \alpha_2^{1-\frac{1}{p}} T_2 \pm \ldots \pm \alpha_n^{1-\frac{1}{p}} T_n\right) x, x\right\rangle\right|.
\end{aligned}$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

Now we give another upper bound for the powers of w_p. This result has several inequalities as special cases, which considerably generalize the second inequality of (1.1).

Theorem 3.2. *Let* $(T_1, \ldots, T_n), (A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathbb{B}(\mathscr{H})^{(n)}$ *and let* f *and* g *be nonnegative continuous functions on* $[0, \infty)$ *satisfying* $f(t)\,g(t) = t$ *for all* $t \in [0, \infty)$. *Then*

$$w_p^{rp}(A_1^* T_1 B_1, \ldots, A_n^* T_n B_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} \Big(\left[B_i^* f^2(|T_i|)\, B_i\right]^{rp} + \left[A_i^* g^2(|T_i^*|)\, A_i\right]^{rp} \Big) \right\|$$

for $p \geq 1$ *and* $r \geq 1$.

Proof. Let $x \in \mathscr{H}$ be a unit vector.

$$\begin{aligned}
&\sum_{i=1}^{n} |\langle A_i^* T_i B_i x, x\rangle|^p \\
&\quad = \sum_{i=1}^{n} |\langle T_i B_i x, A_i x\rangle|^p \\
&\quad \leq \sum_{i=1}^{n} \|f(|T_i|)\, B_i x\|^p \, \|g(|T_i^*|)\, A_i x\|^p \qquad \text{(by Lemma 2.1}(c)) \\
&\quad = \sum_{i=1}^{n} \langle f(|T_i|)\, B_i x, f(|T_i|)\, B_i x\rangle^{\frac{p}{2}} \langle g(|T_i^*|)\, A_i x, g(|T_i^*|)\, A_i x\rangle^{\frac{p}{2}} \\
&\quad = \sum_{i=1}^{n} \left\langle B_i^* f^2(|T_i|)\, B_i x, x\right\rangle^{\frac{p}{2}} \left\langle A_i^* g^2(|T_i^*|)\, A_i x, x\right\rangle^{\frac{p}{2}} \\
&\quad \leq \sum_{i=1}^{n} \left\langle \left(B_i^* f^2(|T_i|)\, B_i\right)^p x, x\right\rangle^{\frac{1}{2}} \left\langle \left(A_i^* g^2(|T_i^*|)\, A_i\right)^p x, x\right\rangle^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.2}(a)) \\
&\quad \leq \sum_{i=1}^{n} \left(\frac{1}{2} \left(\left\langle \left(B_i^* f^2(|T_i|)\, B_i\right)^p x, x\right\rangle^r + \left\langle \left(A_i^* g^2(|T_i^*|)\, A_i\right)^p x, x\right\rangle^r \right) \right)^{\frac{1}{r}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1}(a)) \\
&\quad \leq \sum_{i=1}^{n} \left(\frac{1}{2} \left\langle \left(\left(B_i^* f^2(|T_i|)\, B_i\right)^{rp} + \left(A_i^* g^2(|T_i^*|)\, A_i\right)^{rp} \right) x, x\right\rangle \right)^{\frac{1}{r}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.2}(a)) \\
&\quad \leq \left(\frac{1}{2} \left\langle \sum_{i=1}^{n} \left(\left(\left(B_i^* f^2(|T_i|)\, B_i\right)^{rp} + \left(A_i^* g^2(|T_i^*|)\, A_i\right)^{rp} \right) \right) x, x \right\rangle \right)^{\frac{1}{r}}
\end{aligned}$$

Thus

$$\left(\sum_{i=1}^{n}|\langle A_i^*T_iB_ix,x\rangle|^p\right)^r$$
$$\leq\frac{1}{2}\left\langle\left(\sum_{i=1}^{n}\left(\left(B_i^*f^2\left(|T_i|\right)B_i\right)^{rp}+\left(A_i^*g^2\left(|T_i^*|\right)A_i\right)^{rp}\right)\right)x,x\right\rangle$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. $\square$

Choosing $A=B=I$, we get.

Corollary 3.3. *Let $(T_1,\ldots,T_n)\in\mathbb{B}(\mathscr{H})^{(n)}$ and let f and g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)\,g(t)=t$ for all $t\in[0,\infty)$. Then*

$$w_p^{rp}\left(T_1,\ldots,T_n\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}\left(f^{2rp}\left(|T_i|\right)+g^{2rp}\left(|T_i^*|\right)\right)\right\|$$

for $p\geq1$ and $r\geq1$.

Letting $f(t)=g(t)=t^{\frac{1}{2}}$, we get.

Corollary 3.4. *Let $(T_1,\ldots,T_n),(A_1,\ldots,A_n),(B_1,\ldots,B_n)$ are in $\mathbb{B}(\mathscr{H})^{(n)}$. Then*

$$w_p^{rp}\left(A_1^*T_1B_1,\ldots,A_n^*T_nB_n\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}\Big(\left(B_i^*\left|T_i\right|B_i\right)^{rp}+\left(A_i^*\left|T_i^*\right|A_i\right)^{rp}\Big)\right\|$$

for $p\geq1$ and $r\geq1$.

Corollary 3.5. *Let $(A_1,\ldots,A_n),(B_1,\ldots,B_n)\in\mathbb{B}(\mathscr{H})^{(n)}$. Then*

$$w_p^{rp}\left(A_1^*B_1,\ldots,A_n^*B_n\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}\Big(\left|B_i\right|^{2rp}+\left|A_i\right|^{2rp}\Big)\right\|$$

for $p\geq1$ and $r\geq1$.

Corollary 3.6. *Let $(T_1,\ldots,T_n)\in\mathbb{B}(\mathscr{H})^{(n)}$. Then*

$$w_p^{p}\left(T_1,\ldots,T_n\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}\left(\left|T_i\right|^{2\alpha p}+\left|T_i^*\right|^{2(1-\alpha)p}\right)\right\|$$

for $0\leq\alpha\leq1$, and $p\geq1$. In particular.

$$w_p^{p}\left(T_1,\ldots,T_n\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}\left(|T_i|^p+|T_i^*|^p\right)\right\|.$$

Corollary 3.7. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then*

$$w_p^p(B,C) \leq \frac{1}{2}\left\| \, |B|^{2\alpha p} + |B^*|^{2(1-\alpha)p} + |C|^{2\alpha p} + |C^*|^{2(1-\alpha)p} \right\|$$

for $0 \leq \alpha \leq 1$, and $p \geq 1$. In particular.

$$w_p^p(B,C) \leq \frac{1}{2}\left\| |B|^p + |B^*|^p + |C|^p + |C^*|^p \right\|.$$

The next results are related to some different upper bounds for w_p for n-tuples of operators, which have several inequalities as special cases.

Proposition 3.8. *Let $(T_1, \ldots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$. Then*

$$w_p(T_1, \ldots, T_n) \leq \frac{1}{2}\left\| \sum_{i=1}^{n} \left(|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} \right)^p \right\|^{\frac{1}{p}}$$

for $0 \leq \alpha \leq 1$, and $p \geq 1$.

Proof. By using the arithmetic-geometric mean, for any unit vector $x \in \mathscr{H}$ we have

$$\begin{aligned}
\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p &\leq \sum_{i=1}^{n} \left(\left\langle |T_i|^{2\alpha} x, x \right\rangle^{\frac{1}{2}} \left\langle |T_i^*|^{2(1-\alpha)} x, x \right\rangle^{\frac{1}{2}} \right)^p \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.1}(b)\text{)} \\
&\leq \frac{1}{2^p}\sum_{i=1}^{n} \left(\left\langle |T_i|^{2\alpha} x, x \right\rangle + \left\langle |T_i^*|^{2(1-\alpha)} x, x \right\rangle \right)^p \\
&= \frac{1}{2^p}\sum_{i=1}^{n} \left\langle \left(|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} \right) x, x \right\rangle^p. \\
&\leq \frac{1}{2^p}\sum_{i=1}^{n} \left\langle \left(|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)} \right)^p x, x \right\rangle \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.2}(a)\text{)}
\end{aligned}$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

Proposition 3.9. *Let $(T_1, \ldots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$. Then*

$$w_p(T_1, \ldots, T_n) \leq \left\| \sum_{i=1}^{n} \left(\alpha |T_i|^p + (1-\alpha) |T_i^*|^p \right) \right\|^{\frac{1}{p}}$$

for $0 \leq \alpha \leq 1$, and $p \geq 2$.

Proof. For every unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p \\
&= \sum_{i=1}^{n} \left(|\langle T_i x, x\rangle|^2\right)^{\frac{p}{2}} \\
&\le \sum_{i=1}^{n} \left(\left\langle |T_i|^{2\alpha} x, x\right\rangle \left\langle |T_i^*|^{2(1-\alpha)} x, x\right\rangle\right)^{\frac{p}{2}} \qquad \text{(by Lemma 2.1}(b)) \\
&\le \sum_{i=1}^{n} \left\langle |T_i|^{\alpha p} x, x\right\rangle \left\langle |T_i^*|^{(1-\alpha)p} x, x\right\rangle \qquad \text{(by Lemma 2.2}(a)) \\
&\le \sum_{i=1}^{n} \left\langle |T_i|^{p} x, x\right\rangle^{\alpha} \left\langle |T_i^*|^{p} x, x\right\rangle^{(1-\alpha)} \qquad \text{(by Lemma 2.2}(b)) \\
&\le \sum_{i=1}^{n} \Big(\alpha \left\langle |T_i|^{p} x, x\right\rangle + (1-\alpha) \left\langle |T_i^*|^{p} x, x\right\rangle\Big) \text{(by Lemma 2.1}(a)) \\
&\le \sum_{i=1}^{n} \Big\langle \Big(\alpha |T_i|^p + (1-\alpha) |T_i^*|^p\Big) x, x\Big\rangle \\
&= \left\langle \left(\sum_{i=1}^{n} \left(\alpha |T_i|^p + (1-\alpha) |T_i^*|^p\right)\right) x, x\right\rangle.
\end{aligned}$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

Remark 3.10. As special cases,

(1) For $\alpha = \frac{1}{2}$, we have

$$w_p^p(T_1, \ldots, T_n) \le \frac{1}{2} \left\| \sum_{i=1}^{n} \left(|T_i|^p + |T_i^*|^p\right) \right\|.$$

(2) For $B, C \in \mathbb{B}(\mathscr{H}), 0 \le \alpha \le 1$, and $p \ge 1$, we have

$$w_p^p(B, C) \le \|\alpha |B|^p + (1-\alpha) |B^*|^p + \alpha |C|^p + (1-\alpha) |C^*|^p\|.$$

In particular,

$$w_p^p(B, C) \le \frac{1}{2} \||B|^p + |B^*|^p + |C|^p + |C^*|^p\|.$$

The next result reads as follows.

Proposition 3.11. *Let $(T_1, \ldots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}, 0 \le \alpha \le 1, r \ge 1$ and $p \ge 1$. Then*

$$w_p(T_1, \ldots, T_n) \le \left(\sum_{i=1}^{n} \left\|\alpha |T_i|^{2r} + (1-\alpha) |T_i^*|^{2r}\right\|^{\frac{p}{2r}}\right)^{\frac{1}{p}}.$$

Proof. Let $x \in \mathscr{H}$ be a unit vector.

$$
\begin{aligned}
&\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p \\
&= \sum_{i=1}^{n} \left(|\langle T_i x, x\rangle|^2\right)^{\frac{p}{2}} \\
&\leq \sum_{i=1}^{n} \left(\left\langle |T_i|^{2\alpha} x, x\right\rangle \left\langle |T_i^*|^{2(1-\alpha)} x, x\right\rangle\right)^{\frac{p}{2}} \qquad \text{(by Lemma 2.1}(b)) \\
&\leq \sum_{i=1}^{n} \left(\left\langle |T_i|^2 x, x\right\rangle^{\alpha} \left\langle |T_i^*|^2 x, x\right\rangle^{(1-\alpha)}\right)^{\frac{p}{2}} \qquad \text{(by Lemma 2.2}(b)) \\
&\leq \sum_{i=1}^{n} \left(\alpha \left\langle |T_i|^2 x, x\right\rangle^{r} + (1-\alpha)\left\langle |T_i^*|^2 x, x\right\rangle^{r}\right)^{\frac{p}{2r}} \text{ (by Lemma 2.1}(a)) \\
&\leq \sum_{i=1}^{n} \left(\alpha \left\langle |T_i|^{2r} x, x\right\rangle + (1-\alpha)\left\langle |T_i^*|^{2r} x, x\right\rangle\right)^{\frac{p}{2r}} \text{(by Lemma 2.2}(a)) \\
&\leq \sum_{i=1}^{n} \left\langle \left(\alpha |T_i|^{2r} + (1-\alpha)|T_i^*|^{2r}\right) x, x\right\rangle^{\frac{p}{2r}}.
\end{aligned}
$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

Remark 3.12. Some special cases can be stated as follows:

(1) For $\alpha = \frac{1}{2}$, we have

$$
w_p(T_1, \ldots, T_n) \leq \left(\frac{1}{2^{\frac{p}{2r}}} \sum_{i=1}^{n} \left\| |T_i|^{2r} + |T_i^*|^{2r} \right\|^{\frac{p}{2r}}\right)^{\frac{1}{p}}.
$$

(2) For $B, C \in \mathbb{B}(\mathscr{H}), 0 \leq \alpha \leq 1$, and $p \geq 1$, we have

$$
\begin{aligned}
&w_p(B, C) \\
&\leq \left(\left\|\alpha |B|^{2r} + (1-\alpha)|B^*|^{2r}\right\|^{\frac{p}{2r}} + \left\|\alpha |C|^{2r} + (1-\alpha)|C^*|^{2r}\right\|^{\frac{p}{2r}}\right)^{\frac{1}{p}}.
\end{aligned}
$$

In particular,

$$
w_p(B, C) \leq \frac{1}{2^{\frac{1}{2r}}} \left(\left\||B|^{2r} + |B^*|^{2r}\right\|^{\frac{p}{2r}} + \left\||C|^{2r} + |C^*|^{2r}\right\|^{\frac{p}{2r}}\right)^{\frac{1}{p}}.
$$

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